In this chapter we will explore the applications of differentiation and integration, mainly to solve differential equations. An application of differential Equations is that they could be used to model the population infected by maybe a **global pandemic (they were used in coronavirus),** so basically, they are very important.

1) **First Order linear differential equations**
2) **Second-order homogeneous differential equations**
3) **Second-order non-homogeneous differential equations**
4) **Using boundary conditions**

Prerequisite work- Simple first differential equations

So far you may have covered simple first order differential equations. Solve the following below and find the general solution to the differential equation (a) and particular solution to differential equation (b)

a) $\frac{dy}{dx} = xe^x$

b) $\frac{dy}{dx} = -\frac{x}{y}$ given that y=0 when x=2

First order Differential equation

A first order differential equation is where the highest order of the derivative is 1 equal. There are 2 ways of solving a first order differential equation.

1) Separation of variables

In general, if **dy/dx = f(x)g(y)** then we can separate the x and y variables

⇒ $\int \frac{1}{g(y)} dy = \int f(x)\, dx$ (we have separated the variables)

Example (Separation of variables)

If $\frac{dy}{dx} = -xy$

$\int \frac{1}{y} dy = \int -x\, dx$

$ln|y| = -\frac{1}{2}x^{2+c}$

$y = e^{-\frac{1}{2}x^2+c}$

$y = e^{-\frac{1}{2}x^2}e^c$

$y = Ae^{-\frac{1}{2}x^2}$ (this is our final general solution)

Pupil question

$(x+1)\frac{dy}{dx} - y = 0$

2) Reverse product rule

Exact first order differential equations in the form:

$$\mathbf{f(x)\frac{dy}{dx} + f'(x)y = g(x)} \quad \textbf{(1)}$$

Using our mathematical intuition and previous mathematical tools we can say:

$\frac{dy}{dx}\,[f(x)y] = f(x)\frac{dy}{dx} + f'(x)y$ (by use of product rule). (2)

∴ from the product rule we can substitute the LHS of the above equation (2) into the LHS of equation (1)

$\Rightarrow \frac{dy}{dx}\,[f(x)y] = g(x)$

= $\int [f(x)y]\,\frac{d}{dx} = \int g(x)\,dx$

Example

$x^4\frac{dy}{dx} + 4x^3y = e^{2x}$ (use reverse product rule on the LHS and think what function differentiated gave LHS)

$\frac{dy}{dx}\,[x^4y] = x^4\frac{dy}{dx} + 4x^3y$

$\therefore \frac{dy}{dx}\,[x^4y] = e^{2x}$

$[x^4y] = \int e^{2x}\,dx$

$x^4y = \frac{1}{2}e^{-2x} + c$

Pupil Question

$x^3\frac{dy}{dx} + 3x^2y = sinx$

Using an integrating factor

Sometimes our first order differential equation is not in our exact form and we have to turn our differential equation into an exact form, shown in equation (1) above.

If our first order differential equation is in the form below use a particular Integral:

$$\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{P(x)y} = \mathbf{Q(x)}.$$ **Where p and q are functions of x**

e.g . $\frac{\text{dy}}{\text{dx}} - 5\text{y} = e^x$. (we can see there is no reverse product of the LHS)

∴ we can turn our first order differential equation into an exact first order differential equation shown on slide 3 by using an integrating factor.
(the proof is very extensive so email if you would like to see it).

The integrating factor is known as:

$\mathbf{e^{\int p(x)dx}}$ (we multiply each term in our differential equation by this integrating factor, where p(x) is the function multiplied by y, will turn our DE into an exact form).

Example (Using Integrating factor)

$\frac{\text{dy}}{\text{dx}} - 4\text{y} = e^x$ (Step 1: recognize the form of the differential equation)

$\text{e}^{\int -4\text{dx}} = e^{-4x}$ (Step 2: find the integrating factor, p(x) = -4)

$e^{-4x}\frac{\text{dy}}{\text{dx}} - 4e^{-4x}\text{y} = e^x e^{-4x}$ (Step 3: Multiply each term by the integrating factor, **now in an exact form**)

$[\text{e}^{-4x}y] = \int \text{e}^{-3x}\, dx$ (Step 4: Use the reverse product rule)

$[\text{e}^{-4x}y] = \frac{-1}{3}e^{-3x} + c$ (Step 5: Integrate)

$y = \frac{-1}{3}e^x + c\, e^{4x}$ (Step 6: write in form y=)

Pupil Question

Find the general solution of the differential equation

$$cosx\frac{dy}{dx} + 2ysinx = cos^4x$$

Second Order homogenous differential equations

A second order homogenous equation is one that's highest order of derivative is 2. It becomes homogenous when it is equal to zero. Shown in the form below

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \text{ where } a, b\ c \text{ are consants}$$

When solving a second order differential equation we try to find a general solution to our equation that can help us solve our 2nd ODE.

Mathematical Intuition

e.g. Lets say our 2nd ODE is:

$\frac{d^2y}{dx^2} - 4y = 0$ ($where\ a = 1, b = 0\ and\ c = -4$)

$\frac{d^2y}{dx^2} = 4y$

This says the 2nd derivative is 4 times the original function, we need to think what function when we differentiate it gives us the original function back.

$\Rightarrow e^{2x}$ (An exponential functional when differentiated gives us the original function back)

Let y= e^{2x}

Y'= $2e^{2x}$

Y'' = $4e^{2x}$ (so the second derivative is 4 times the original function)

So we can say y(x) = Ae^{rx} is a solution to our general 2nd order homogenous differential equation.

Example

Y'' + 5y' +6y' = 0 (we know Ae^{rx} when differentiated gives some constant out the front time the original function)

∴ Lets assume y= Ae^{rx} is a solution to our ODE where r is to be found, by subbing in for Ae^{rx}

Let y= Ae^{rx}
Y'= Are^{rx}
Y'' = Ar^2e^{rx}

r^2e^{rx} + $5re^{rx}$ + $6e^{rx}$ =0

Ae^{rx} (r^2+5r+6)=0 (here Ae^{rx} cannot be zero therefore we have to solve the quadratic for r)

r^2+5r+6=0 (this here is called the **auxiliary equation)**

(r+2)(r+3)=0

r=-2 or r=-3. (real roots)

so our general solution to our 2nd homogenous ODE becomes:

y(x) = Ae^{-2x} + $Be^{-3x.}$ (where A and B our constants)

<u>3 Cases of our auxiliary equation (ar^2+b^r+c) =0</u>

Each of these cases give 3 different general solutions shown each equation. Each of the differential equations can be shown, try to prove these results yourself.

1) When b^2-4ac > 0 (real roots)

$$\mathbf{y = Ae^{\alpha x} + Be^{\beta x}}$$

2) When b^2-4ac = 0 (repeated roots)

$$\mathbf{y = (A + Bx)^{\alpha x}}$$

3) When b^2-4ac < 0 (complex roots, p±qi)

$$y = \mathbf{e^{px}}(\boldsymbol{A}\mathbf{cos(qx) + Bsin(qx)})$$

Pupil Question (Repeated Roots)

a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Pupil Question (Complex Roots)

b) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$

Second Order non- homogenous differential equations

A second order non- homogenous equation is one that's highest order of derivative is 2. It becomes non-homogenous when it is equal to some function of x. General form shown below:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \text{ where a, b c are consants}$$

1)To solve an equation of this type we first have to consider the homogenous form and find the general solution this is called the **complementary function.**

2) Then we need to find a **particular integral** which is a function that satisfies the RHS.

To find a particular integral we use the table below:

Form of f(x)- RHS of DE	Form of Particular Integral
p	λ
p+qx	$\lambda + \mu x$
$p+qx+rx^2$	$\lambda + \mu x + vx^2$
pe^{kx}	λe^{kx}
pcos(wx) + qsin(wx)	$\lambda \cos(wx) + \mu\sin(wx)$

Example (no homogeneous differential equations)

Find the general solution to the following differential equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2x$$

1) Firstly, Solve the Homogenous form and find the complementary function.

Jumping straight to our Auxiliary equation:

$r^2-5r+6=0$ (Auxiliary equation)

$(r-2)(r-3)=0$

r=2 or r=3 (real roots)

so our complementary function becomes

$y(x) = Ae^{2x} + Be^{3x}$ (where A and B our constants)

2) Second, Find the Particular Integral

From our table we can see our particular integral is in the form p+qx (where p = 0 and q =2).

$\therefore Let\ y = \lambda + \mu x$ (Particular Integral)

$\frac{\mathrm{dy}}{\mathrm{dx}} = \mu$

$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = 0$

We have differentiated our particular Integral form, which we substitute into our differential equation.

$\mathbf{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2x}$

$0 - 5\mu + 6(\lambda + \mu x) = 2x$

$-5\mu + 6\lambda + 6\mu x = 2x.$

$6\lambda - 5\mu + 6\mu x = 2x$ (from here we compare the LHS with the RHS)

$6\lambda - 5\mu = 0 \ \ or\ 6\mu x = 2x$ (solve for $\lambda\ and\ \mu$)

$\therefore \mu = \frac{1}{3}\ and\ \lambda = \frac{5}{18}$

So our particular Integral is

$\frac{1}{3}x + \frac{5}{18}$

Our overall general solution to our 2^{nd} ODE is:

$y(x) = Ae^{2x} + Be^{3x} + \frac{1}{3}x + \frac{5}{18}$ (y=CF +PI)

Pupil Question

Find the particular Integral for the differential equation

$\mathbf{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)}$ where f(x) is:

a) 3

b) e^x

c) 13sin3x

A few caveats when choosing a particular Integral- Special cases

Example 1

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

If we begin to solve this as normal

m^2 -5m +6y =0

(m-2)(m-3) = 0

m=2 or m=3

C.F $y(x) = Ae^{2x} + Be^{3x}$

P.I. $y(x) = \lambda e^{2x}$

New P.I $y(x) = \lambda x e^{2x}$

Normally we use this form for our particular Integral however we cannot use a particular integral that overlaps with our complementary function as both are in the same form of e^{2x}; therefore, we multiply it by x.

Complete this example below using new particular integral.

Example 2

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$$

$m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0$ or $m = 2$

C.F $y(x) = A + Be^{2x}$

New P.I $y(x) = \lambda x$

Normally we use this form for our particular Integral however we cannot use a particular integral that overlaps with our complementary function as both are constants λ *and* A; therefore, we multiply it by x.

Complete this example below using the new particular integral.

Using boundary conditions

You can use given boundary conditions to find a particular solution to a second-order differential equation. Since there are two arbitrary constants, you'll need two boundary conditions to determine the complete particular solution.

Example

Find in terms of x, given $\frac{d^2y}{dx^2} - y = 2e^x$, and that $\frac{dy}{dx} = 0$ and y= 0 at x=0.

<u>Example</u>

Given that a particular integral is of the form $\lambda sin2t$, find the solution to the differential equation

$\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{d}t} - \mathrm{x} = 3\sin 2\mathrm{t}$, for which x=0 and $\frac{dx}{dt^2} = 1$ when t=0.

Summary of key points

Summary of key points

1 You can solve a first-order differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ by multiplying every term by the **integrating factor** $e^{\int P(x)dx}$.

2 The natures of the roots α and β of the **auxiliary equation** determine the **general solution** to the second-order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0$.
You need to consider three different cases:

- **Case 1: $b^2 > 4ac$**
  The auxiliary equation has two real roots α and β ($\alpha \neq \beta$). The general solution will be of the form $y = Ae^{\alpha x} + Be^{\beta x}$ where A and B are arbitrary constants.
- **Case 2: $b^2 = 4ac$**
  The auxiliary equation has one repeated root α. The general solution will be of the form $y = (A + Bx)e^{\alpha x}$ where A and B are arbitrary constants.
- **Case 3: $b^2 < 4ac$**
  The auxiliary equation has two complex conjugate roots α and β equal to $p \pm qi$. The general solution will be of the form $y = e^{px}(A\cos qx + B\sin qx)$ where A and B are arbitrary constants.

3 A **particular integral** is a function which satisfies the original differential equation.

4 To find the general solution to the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$,

- Solve the corresponding homogeneous equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ to find the complementary function, C.F.
- Choose an appropriate form for the particular integral, P.I., and substitute into the original equation to find the values of any coefficients.
- The general solution is $y = \text{C.F.} + \text{P.I.}$