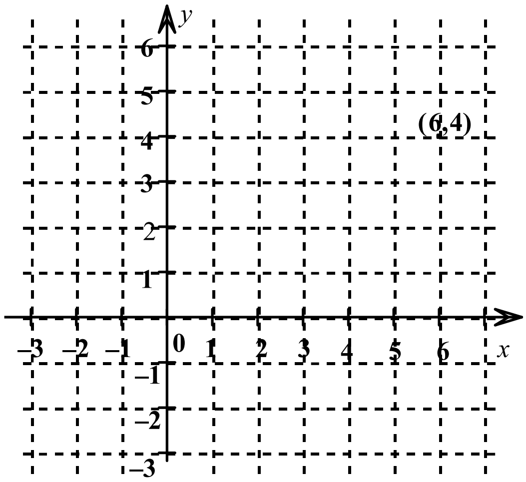
## Converting Cartesian Coordinates to Polar Coordinates




**Cartesian Coordinates**
(x,y) = (6,4)

**Polar Coordinates**
( r , θ) = ( , )

Modulus/
Direct distance
From the origin

Angle between
the positive
x axis and the
modulus line
**in radians**

We can use trigonometry to convert Cartesian Coordinates to Polar Coordinates or we can use the following formulae.

$r^2 = x^2 + y^2$ ---------------- **FORMULA 1**

$\theta = \arctan\left(\frac{y}{x}\right)$ ---------------- **FORMULA 2**

**NOTE:** When using this formula, you may need to adapt you answer so θ lies in the correct quadrant

**Examples (Same as Page 101)**

Find the polar coordinates of the following points:

a) (3,4) b) (5, -12) c( $-\sqrt{3}$ , -1)

Use the

$r^2 = x^2 + y^2$

$\theta = \arctan\left(\frac{y}{x}\right)$

formulae!

## Converting Polar Coordinates to Cartesian Coordinates

We can use trigonometry to convert Polar Coordinates to Cartesian Coordinates or we can use the following formulae.

**$r \cos \theta = x$ ---------------- FORMULA 3**

**$r \sin \theta = y$ ---------------- FORMULA 4**

**Examples (Same as Page 102)**

Find the cartesian coordinates of the following points:

a) $\left(10\ ,\ \frac{4\Pi}{3}\right)$

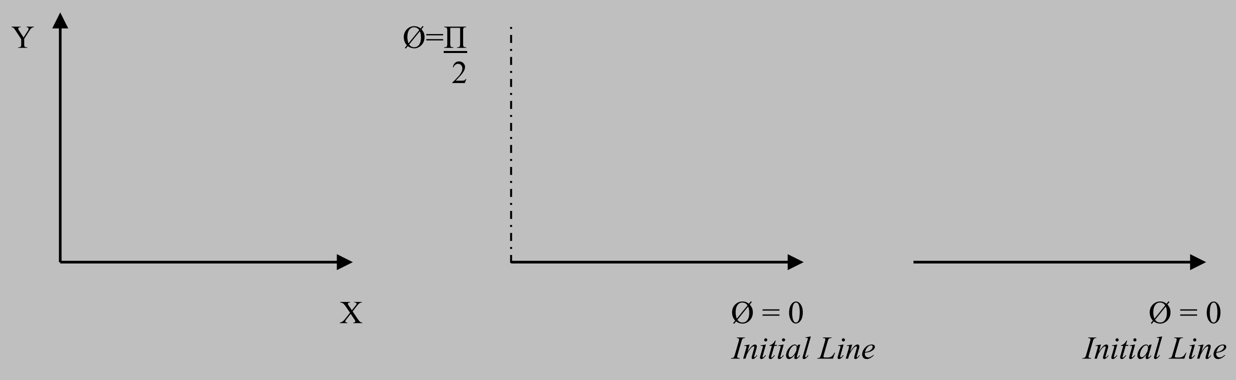
b) $\left(8\ ,\ \frac{2\Pi}{3}\right)$

Use the

**$r \cos \theta = x$**
**$r \sin \theta = y$**

formulae!

**Polar Coordinate Axes**




(Normally these axes are used for cartesian coordinates /problems)

(Either of these axes tend to be used for polar coordinates or problems)

## A Level Mathematics – Double Angle Trig Formulae Reminder

**1.** $\sin 2A = 2\sin A \cos A$

**2.** $\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

**3.** $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

By using the trig identity
$\sin^2 A + \cos^2 A = 1$
we can also express cos 2A in the above two formats

### Converting Polar Equations to Cartesian Equations

**Examples (Same as Page 103)**

Find Cartesian Equations for the following

a) $r = 5$

b) $r = 2 + \cos 2\theta$

c) $r^2 = \sin 2\theta \quad 0 < \theta < \dfrac{\pi}{2}$

**Objective**: To replace r with x's and y's

- Manipulate your formula so you have an $r^2$
- Replace $r^2$ using the $\mathbf{r^2 = x^2 + y^2}$ **formula.**

**Objective**: To replace r and cos 2θ with x's and y's

- Manipulate your formula so you have $\mathbf{r^2}$ and / or $\mathbf{r\cos\theta}$ and/or $\mathbf{r\sin\theta}$
  **(Note:** $\mathbf{r^3}$ is also fine since $\mathbf{r^3 = (r^2)^{3/2}}$

  $r^2 = x^2 + y^2$
  $r^3 = (x^2 + y^2)^{3/2}$

- As you have a cos2θ or sin 2θ , use an appropriate double angle formulae to convert into terms of cos θ or sin θ

Replace any $r^2$ using $\mathbf{r^2 = x^2 + y^2}$ and any r cos θ or r sin θ with $\mathbf{r\cos\theta = x}$ $\mathbf{r\sin\theta = y}$

**Converting Polar Equations to Cartesian Equations - Continued**

**Pupil Q's**
Find Cartesian Equations for the following

**g** $r = 4(1 - \cos 2\theta)$ **h** $r = 2\cos^2\theta$ **i** $r^2 = 1 + \tan^2\theta$

**Note:**

**$1 + \tan^2 A = \sec^2 A$**

**Answers:**
**g** $(x^2 + y^2)^{\frac{3}{2}} = 8y^2$ **h** $(x^2 + y^2)^{\frac{3}{2}} = 2x^2$
**i** $x^2 = 1$

---

**Converting Cartesian Equations to Polar Equations**

**Examples (Same as Page 103)**

Find Polar Equations for the following

a) $y^2 = 4x$ b) $x^2 - y^2 = 5$ c) $y\sqrt{3} = x + 4$

Use the

**$r \cos \theta = x$**
**$r \sin \theta = y$**

formulae to substitute for x and y

**Converting Cartesian Equations to Polar Equations - Continued**

**Pupil Questions**

1. Find Polar Equations for the following

**a** $x^2 + y^2 = 16$ **b** $xy = 4$ **c** $(x^2 + y^2)^2 = 2xy$

**2.** Convert $x^2 + (y - 2)^2 = 4$ into its Polar form

**Answers**

**1.**

a $r = 4$ **b** $r^2 = 8\,\mathrm{cosec}\,2\theta$

c $r^2 = \sin 2\theta$ **d** $r = 2\cos\theta$

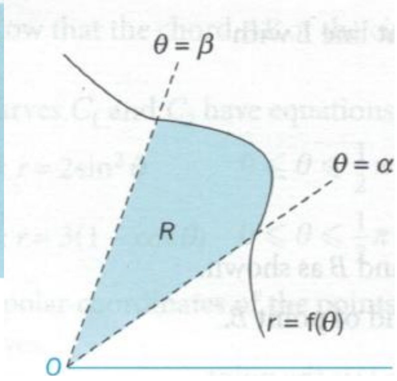
*2.* $r = 4\sin\theta$

**Area of a Sector of a Polar Curve**

The area $A$ of region $R$ bounded by the curve $r = f(\theta)$ and the half-lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

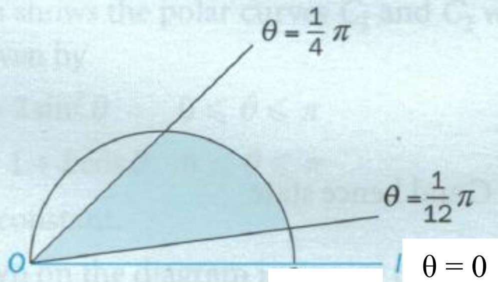
given that $r \geqslant 0$ for all values of $\theta$ where $\alpha \leqslant \theta \leqslant \beta$.




**Example**

The diagram shows a semi-circle with polar equation

$r = 2\cos\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.




**a** Find the area of the region bounded by the curve and the half-lines $\theta = \frac{1}{12}\pi$ and $\theta = \frac{1}{4}\pi$, shaded in the diagram. Give your answer to 2 decimal places.

**b** Use calculus to show that the area of the semi-circle is $\frac{1}{2}\pi$.

**A Level Maths - Integration Reminder**

**Double Angles are easy to integrate, power functions are not!**

$\cos k\theta$ integrates to $\frac{1}{k}\sin k\theta$

$\sin k\theta$ integrates to $-\frac{1}{k}\cos k\theta$

**A Level Maths - Converting Squared Functions into Double Angles**

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

***Answer**: a) 0.77 b) /*

**Area of a Sector of a Polar Curve Continued - Exam Question - (June 2011)**

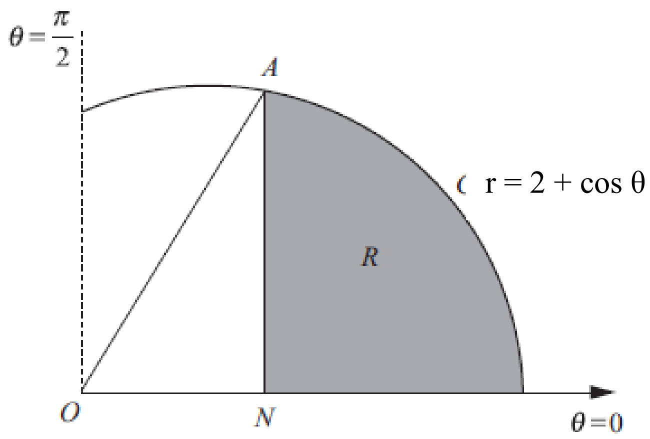



**Figure 1**

The curve *C* shown in Figure 1 has polar equation

$$r = 2 + \cos\theta, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

At the point *A* on *C*, the value of *r* is $\frac{5}{2}$.

The point *N* lies on the initial line and *AN* is perpendicular to the initial line.

The finite region *R*, shown shaded in Figure 1, is bounded by the curve *C*, the initial line and the line *AN*.

Find the exact area of the shaded region *R*.

**(9)**

***Answer***

$$\text{Area of } R = \frac{3\pi}{4} + \frac{17\sqrt{3}}{16} - \frac{25\sqrt{3}}{32} = \boxed{\frac{3\pi}{4} + \frac{9\sqrt{3}}{32}}$$

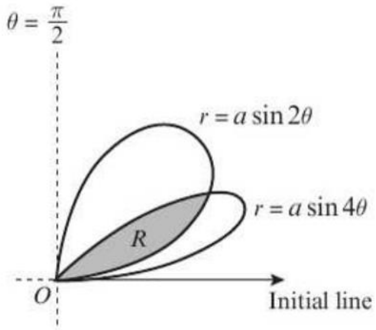
**Area of a Sector of a Polar Curve Continued - Pupil Questions**

**5** The diagram shows the curves with equations $r = a\sin 4\theta$ and $r = a\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

The finite region $R$ is contained within both curves.

Find the area of $R$, giving your answer in terms of $a$.

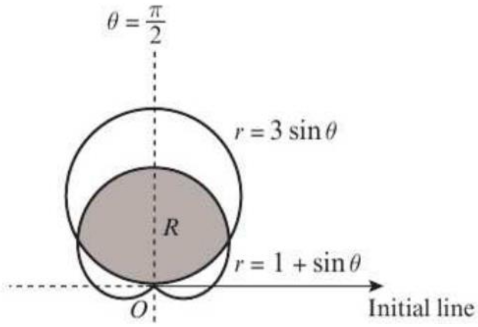
**(8 marks)**




**6** The diagram shows the curves with equations $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

The finite region $R$ is contained within both curves.

Find the area of $R$. **(8 marks)**




***Answers*** **5** $\frac{a^2}{4}\left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right)$

**6** $\frac{5\pi}{4}$

## Intersection of Two Polar Curves

| |
|---|
| Solve the two equations simultaneously to find the point(s) of intersection. |

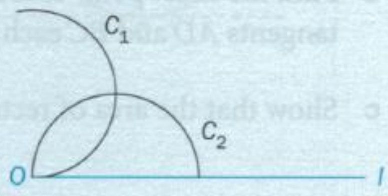
**Example**

The diagram shows the polar curves $C_1$ and $C_2$ with polar equations

$$C_1: r = \sqrt{2}\sin\theta \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$$

and $$C_2: r = \sqrt{2}\cos\theta \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$$

Find the polar coordinates of the point $P$ where the two curves intersect.



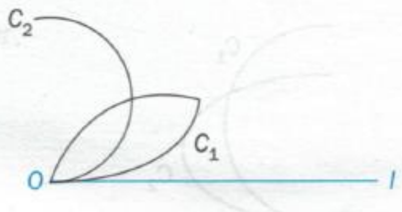

*Answer:* *P( 1, Π/4)*

**Pupil Q1**

The diagram shows the polar curves $C_1$ and $C_2$ with equations given by

$$C_1: r = \sin 2\theta \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$$

and $$C_2: r = \sin\theta \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$$




**a** Verify that the two curves intersect at the pole $O$.

Just verify that (0,0) lies on both curves!

**b** Find the exact coordinates of the other point where these curves meet.

In a trig equation, always try and collect terms on one side and factorise rather than dividing through by a trig term

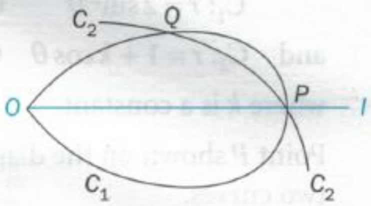
*Answer: b)* ($\sqrt{3}/2$, $\pi/3$)

**Intersection of Two Polar Curves Continued - Pupil Q2**

The diagram shows the polar curves $C_1$ and $C_2$ with equations given by

$C_1$: $r = \cos 2\theta \quad -\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$

and $C_2$: $r = 1 - \sin\theta \quad -\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$




**a** By using a suitable trigonometric identity, find the exact coordinates of the points $P$ and $Q$ where these curves intersect.

**b** Calculate the area of triangle $OPQ$.

Area of a Triangle = ½ a b sin C

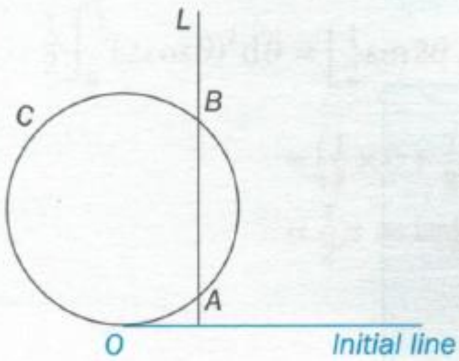
*Answer: a) P (*1, 0), Q(1/2, π/6) b) 1/8

---

**Pupil Q3 – Exam Question**

The diagrams shows the circle $C$ with equation $r = 4\sin\theta$ for $0 \leqslant \theta \leqslant \pi$ and the straight line $L$ with equation $r = \sqrt{3}\sec\theta$ for $0 \leqslant \theta < \frac{1}{2}\pi$.

Line $L$ intersects circle $C$ at points $A$ and $B$.




**a** Find the cartesian equation of the circle $C$ and hence state the radius of $C$.

**b** Find the exact polar co-ordinates of the point $A$ and the point $B$.

**c** Show that line $AB$ has length 2 units.

**d** Find the exact area of the region bounded by the lines $OA$ and $OB$ and the minor arc $AB$ of circle $C$.

To integrate $\sin^2\theta$ *or* $\cos^2\theta$*, you must convert first into* $\cos 2\theta$ *using the appropriate double angle formula in reverse*

Answers

**a** $x^2 + (y-2)^2 = 4$, radius $= 2$

**b** $A\left(2, \frac{1}{6}\pi\right)$, $B\left(2\sqrt{3}, -\frac{1}{3}\pi\right)$

Should be $\pi/3$!

**d** Area $= \frac{2}{3}\pi$

## Tangents to Polar Curves

You need to be able to find tangents to a polar curve that are **parallel** or **perpendicular** to the initial line.

- **To find a tangent parallel to the initial line set $\frac{dy}{d\theta} = 0$.**
- **To find a tangent perpendicular to the initial line set $\frac{dx}{d\theta} = 0$.**

### Example

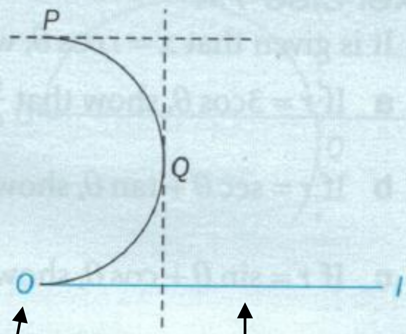
The diagram shows the polar curve with equation $r = \sin\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

The tangents to the curve at points $P$ and $Q$ are, respectively, parallel and perpendicular to the initial line, as shown.

Find the polar coordinates of

**a** point $P$

**b** point $Q$.




In the polar plane, the point 0, is often referred to as the **pole**

The line θ=0, from the pole, is often referred to as the **initial line**

**PARALLEL**

To find the polar points on a curve, C, at which the tangent is parallel to the initial line

- **Write y = r sin θ**
- Replace r using the equation for C
- Find dy/dθ
- Solve dy/dθ = 0, to find **θ**
- Use the equation for C and your value of θ to find **r**

**PERPENDICULAR**

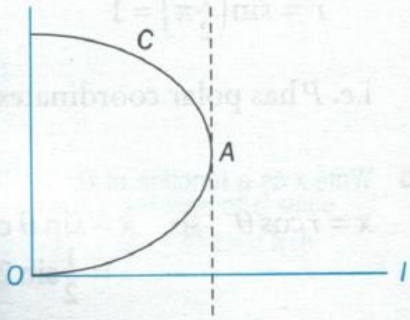
To find the polar points on a curve, C, at which the tangent is perpendicular to the initial line

- **Write x = r cos θ**
- Replace r using the equation for C
- Find dx/dθ
- Solve dx/dθ = 0, to find **θ**
- Use the equation for C and your value of θ to find **r**

Answers: a) (1, Π/2) b) (√2 / 2 , Π/4)

**Tangents to Polar Curves – Pupil Questions**

1 The diagram shows the polar curve with equation
$r = \sin^2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.
The tangent to the curve at point $A$ is perpendicular to the initial line.




**a** Show that the $x$-coordinate of any point $P(r, \theta)$ on the curve is given by $x = \cos\theta - \cos^3\theta$

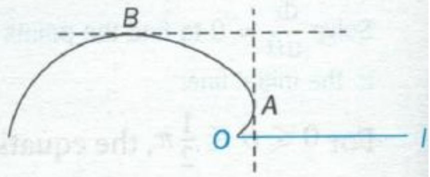
**b** Find $\frac{dx}{d\theta}$.

**c** Hence show that $OA = \frac{2}{3}$

*Answer b)* $-\sin\theta + 3\cos^2\theta\sin\theta$

---

2 The diagram shows the curve with polar equation
$r = 1 - \cos\theta$ for $0 \leqslant \theta \leqslant \pi$.
Tangents to the curve at points $A$ and $B$ are, respectively, perpendicular and parallel to the initial line $l$.




**a** Show that $A$ has polar coordinates $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$.

**b** Find the exact polar coordinates of point $B$ and hence show that the line $AB$ has length $\frac{1}{2}\sqrt{7}$.

*Answer b) B* $(3/2, 2\pi/3)$

---

3. Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$

*(Same as example 12 on page 113, Answer =* $(0, \pi)$ *and* $(3a/2, \pm\pi/3)$*)*

---

4. Find the equations and the points of contact of the tangents to the curve $r = a\sin 2\theta$, $0 \leq \theta \leq \pi/2$ that are

a) parallel to the initial line $\theta = 0$ b) perpendicular to the initial line $\theta = 0$

*(Same as example 12 on page 114, Answer a)* $(0,0)$, $(2a\sqrt{2}/3, 0.955)$ and $r = (4a/(3\sqrt{3}))\operatorname{cosec}\theta$
*b)* $(0, \pi/2)$, $(2a\sqrt{2}/3, 0.615)$ and $r = (4a/(3\sqrt{3}))\sec\theta$*)*

---

**Sketching Polar Equations**

| **Type 1: $r = a$** (Circle, Centre 0, radius a) | **Type 2: $\theta = \alpha$** (Half Line, through 0, making an angle $\alpha$ with the initial line | **Type 3: $r = a\theta$** (Spiral starting at 0) |
|---|---|---|

**Examples (Same as example 5, Page 105)**

1. Sketch $r = 5$

2. Sketch $\theta = \frac{3\Pi}{4}$

3. $r = a\,\theta$ (where a is positive)

---

**Type 4 : no general shape – alternative method needed – Same as Example 6, Page 106**

4. Sketch $r = a\,(1+ \cos\theta)$

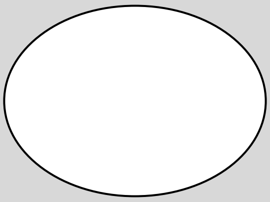
> Draw a table of values for $\theta$ and r
> Try and choose values of $\theta$ where $\cos\theta = -1, 0$ or 1

5. Sketch $r^2 = a^2 \cos 2\theta$

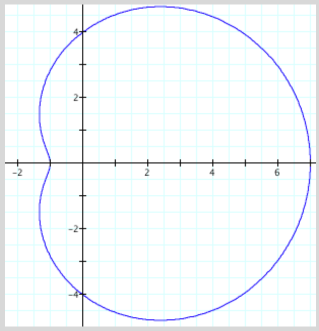
**TYPE5: Curves with equations of the form $r = a\ (p + q \cos \theta)$**

Use the same method as Type 4 but the following knowledge could be very helpful!

The curve will be an egg (when $p \geq 2q$) or a dimple (when $q \leq p < 2q$)



An "egg" is a CONVEX curve



A "dimple" is CONCAVE curve.

The curve is CONCAVE at $\theta = \Pi$

Examples

6. Sketch $r = a(5 + 2 \cos \theta)$

7. Sketch $r = a(3 + 2 \cos \theta)$

## <u>Sketching Polar Equations - Continued</u>

**Pupil Questions**

**1** Sketch the following curves.

**a** $r = 6$ **b** $\theta = \frac{5\pi}{4}$

**d** $r = 2\sec\theta$

**g** $r = a\sin\theta$

**j** $r = a(2 + \cos\theta)$

**o** $r = a(4 + 3\sin\theta)$

**p** $r = 2\theta$

**2**

Sketch the graph with polar equation

$$r = k\sec\left(\frac{\pi}{4} - \theta\right)$$

where $k$ is a positive constant, giving the coordinates of any points of intersection with the coordinate axes in terms of $k$. **(4 marks)**

**3. (Same as example 11, page 110)**

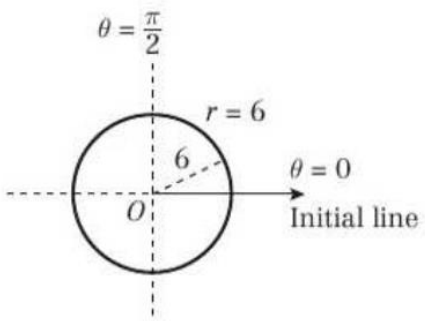
**a** On the same diagram, sketch the curves with equations $r = 2 + \cos\theta$ and $r = 5\cos\theta$.

**b** Find the polar coordinates of the points of intersection of these two curves.

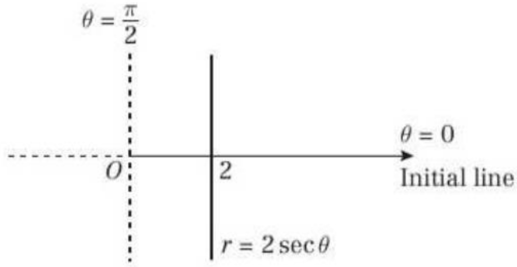
**c** Find the exact area of the region which lies within both curves.

**Answers:**

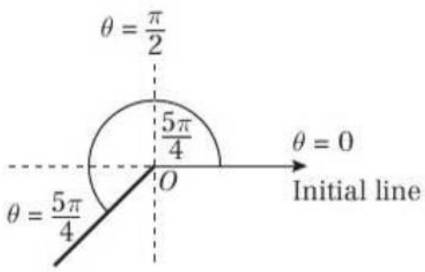
**1 a**



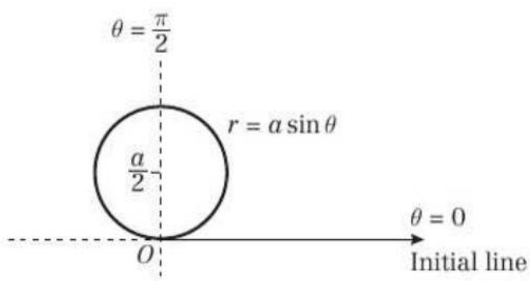

**d**




**b**

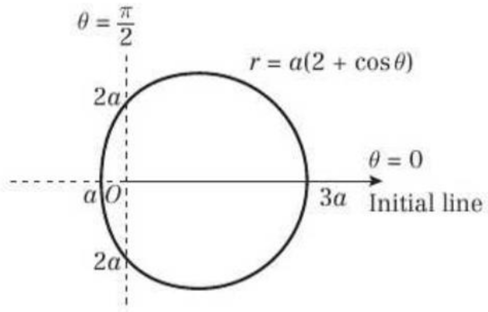



**g**

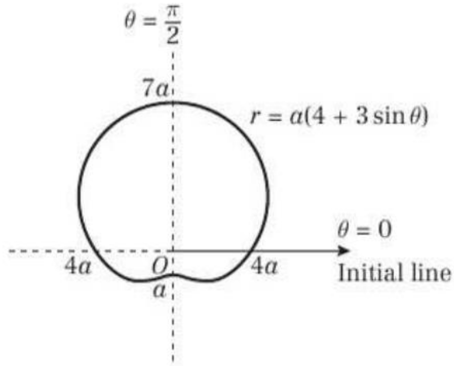


**Answers Continued:**

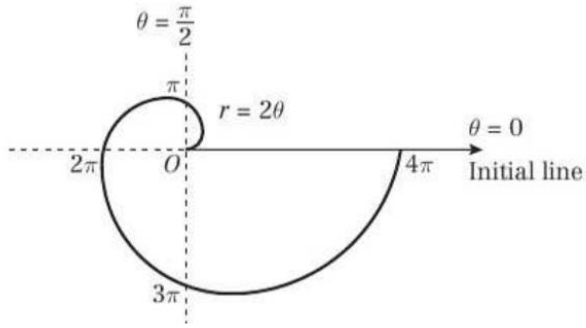
j



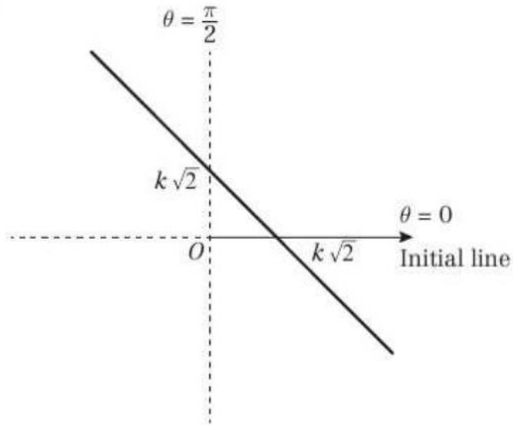

o




p




2




3. a) / b) (5/2, Π/3) (5/2, - Π/3) c) 43Π/12 - ✓3

## Finding a Polar Curve to Represent a Locus of points on an Argand Diagram

### Example

**a** Show on an Argand diagram the locus of points given by the values of $z$ satisfying
$|z - 3 - 4\mathrm{i}| = 5$

**b** Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$.

Method for (a)

- Draw the Locus exactly as you did in the Complex Number Chapter in Year 12

Method for (b)

- Write your Locus in cartesian form (Same as Complex Number Chapter in Year 12)
- Substitute for x and y in polar form

**$r\cos\theta = x$ $\quad$ $r\sin\theta = y$**

- Rearrange into the required format

### Pupil Questions

**3 a** Show on an Argand diagram the locus of points given by the values of $z$ satisfying
$|z - 12 - 5\mathrm{i}| = 13$ **(2 marks)**

**b** Show that this locus of points can be represented by the polar curve
$r = 24\cos\theta + 10\sin\theta$ **(4 marks)**

---

**4 a** Show on an Argand diagram the locus of points given by the values of $z$ satisfying
$|z + 4 + 3\mathrm{i}| = 5$ **(2 marks)**

**b** Show that this locus of points can be represented by the polar curve
$r = -8\cos\theta - 6\sin\theta$ **(4 marks)**

**<u>Finding a Polar Curve to Represent a Locus of points on an Argand Diagram - Continued</u>**

**<u>Pupil Questions - Continued</u>**

**11** **a** Show on an Argand diagram the locus of points given by the values of $z$ satisfying
$|z - 1 - \mathrm{i}| = \sqrt{2}$ **(2 marks)**

**b** Show that this locus of points can be represented by the polar curve
$r = 2\cos\theta + 2\sin\theta$ **(4 marks)**

The set of points, $A$, is defined by

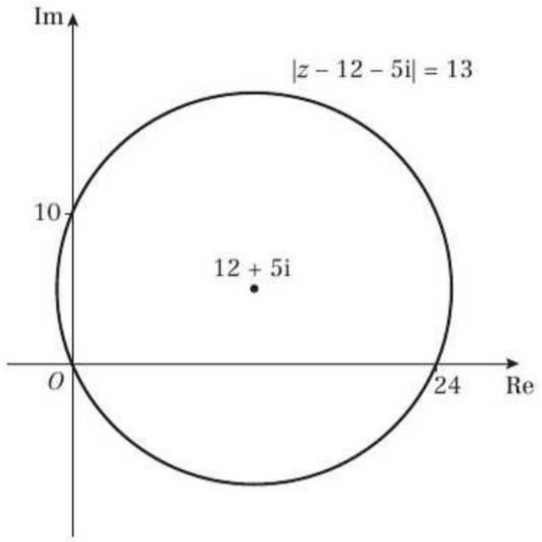
$$A = \left\{z : \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}\right\} \cap \left\{z : |z - 1 - \mathrm{i}| \leqslant \sqrt{2}\right\}$$

**c** Show, by sketching on your Argand diagram, the set of points, $A$. **(2 marks)**

**d** Find, correct to three significant figures, the area of the region defined by $A$. **(5 marks)**

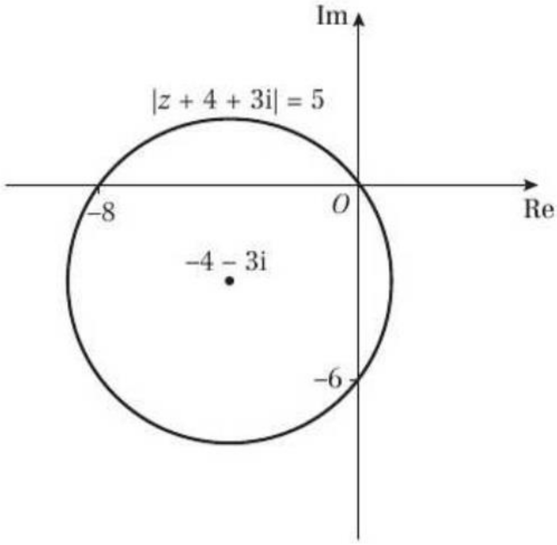
***Answers***

**3 a**



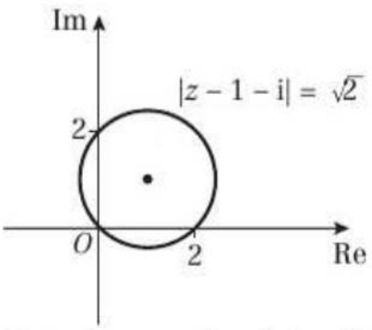

**b** Cartesian equation is $(x - 12)^2 + (y - 5)^2 = 169$
Convert to polar coordinates:
$(r\cos\theta - 12)^2 + (r\sin\theta - 5)^2 = 169$
Then rearrange this to get $r = 24\cos\theta + 10\sin\theta$

**4 a**



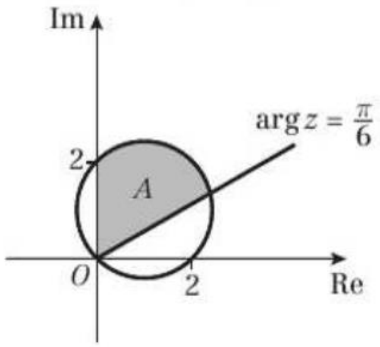

**b** Cartesian equation is $(x + 4)^2 + (y + 3)^2 = 25$
Convert to polar coordinates:
$(r\cos\theta + 4)^2 + (r\sin\theta + 3)^2 = 25$
Then rearrange to get $r = -8\cos\theta - 6\sin\theta$

**11 a**




**b** Cartesian equation is $(x - 1)^2 - (y + 1)^2 = 2$
Convert to polar coordinates:
$(r\cos\theta - 1)^2 + (r\sin\theta - 1)^2 = 2$
Then rearrange to get $r = 2\cos\theta + 2\sin\theta$

**c**




**d** 3.59

**END OF CHAPTER**